Question 2. For $f : X \subset \mathbb{R}^n \to \mathbb{R}$, with coordinates $(x_1, x_2, \ldots, x_i, \ldots, x_n)$ in $\mathbb{R}^n$, do the following:

(b) For a fixed real number $c \in \mathbb{R}$, use the graph of f to describe the c-level set of f. (Do keep in mind that the graph of f and the level sets of f exist in Euclidean spaces of different dimensions.) In your description, discuss the possible sizes, or dimensions of a level set and where the level sets reside. Hint: First think of the case where $n = 2$ and then generalize.

Recall the definition of the c-level set of f. It is the set of all points $\{(x_1, \ldots, x_n)\}$ in the domain such that $f(x_1, \ldots, x_n) = c$. From the definition, it is clear that the level sets live inside the domain $X \subset \mathbb{R}^n$.

To use the graph of f to describe the c-level set of f, consider the following. Let $\Gamma =$

$$\{(x_1, \ldots, x_n, x_{n+1}) \in \mathbb{R}^{n+1} \mid (x_1, \ldots, x_n) \in \overset{\text{domain}}{X},\ x_{n+1} = f(x_1, \ldots, x_n)\}$$

be the graph of f, a subset of $\mathbb{R}^{n+1}$.

Let S be the subset of Γ corresponding to $x_{n+1} = c$.

i.e. $S = \{(x_1, \ldots, x_n, x_{n+1}) \in \Gamma \mid x_{n+1} = f(x_1, \ldots, x_n) = c\}$

Now consider the projection map $\pi : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^n$ given by $\pi(x_1, \ldots, x_n, x_{n+1}) = (x_1, \ldots, x_n)$

Consider the restriction of π to S. Then the pts in the image are precisely those pts of the domain that map to c under f. In other words, the c-level set.

The level set can potentially be a subset of any size in the domain, ranging from the empty set to all of X, depending on f & the value of c. Here are a few examples:

- Let $f: \mathbb{R}^n \to \mathbb{R}$
  $(x_1, \dots, x_n) \mapsto 1$

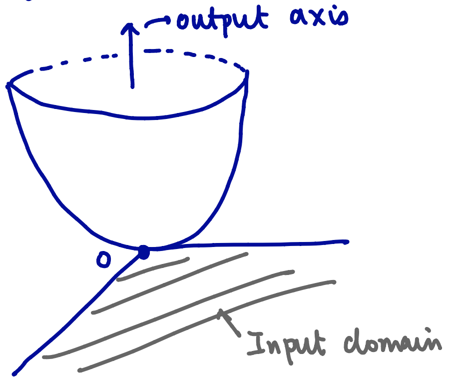
  Then the 1-level set of f is all of the domain $= \mathbb{R}^n$

  The 0-level set of f is $\emptyset$ as there is no point in the domain that maps to 0.

- Let $f: \mathbb{R}^n \to \mathbb{R}$
  $(x_1, \dots, x_n) \mapsto x_n$

  Then the 1-level set of f is all the points in $\mathbb{R}^n$ of the form $(x_1, \dots, x_{n-1}, 1)$. Such points form an $n-1$ dimensional subset as we can choose $n-1$ coordinates freely.

- Let $f: \mathbb{R}^2 \to \mathbb{R}$ described by the following graph



The 0-level set is precisely 1 pt in the domain, the origin as only the origin maps to 0

More generally, to get a level set of any desired size, consider a function that takes a fixed value 0 on a subset of the domain of the desired size & takes a different value

1 on every other point. Then the 0-level set is exactly the size you wanted.

(c) For a fixed real number $a \in \mathbb{R}$, use the graph of f to describe the $(x_i = a)$-vertical slice of the graph of f (the book, on page 80, calls these slices *sections*.) In your description, discuss the possible sizes or dimensions of this slice, where is resides, and how it is situated with regard to the graph of f.

As before, let the graph of f be notated as

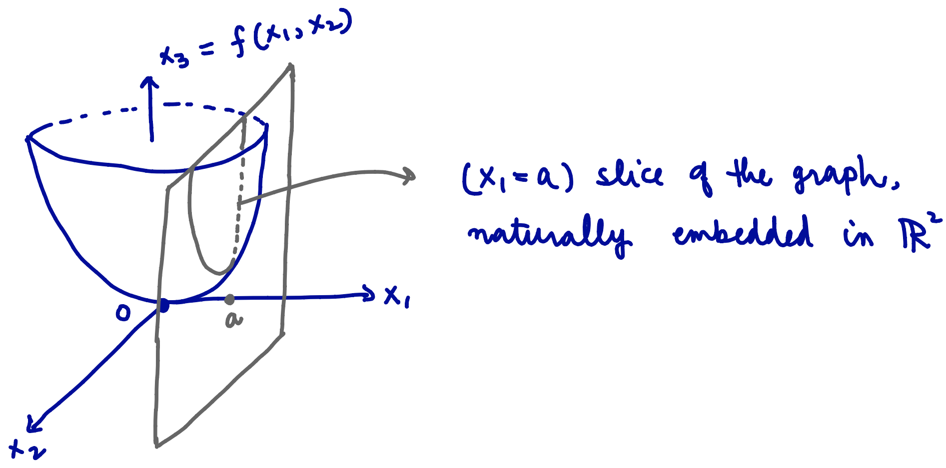
$$\Gamma = \{ (x_1, \ldots, x_n, f(x_1, \ldots, x_n)) \in \mathbb{R}^{n+1} \mid (x_1, \ldots, x_n) \in X \}$$

The $(x_i = a)$-slice is precisely the subset of Γ given by

$$\{ (x_1, \ldots x_{i-1}, \underset{\uparrow \text{ ith position}}{a}, x_{i+1}, \ldots, x_n, f(x_1, \ldots, a, \ldots x_n)) \in \mathbb{R}^{n+1} \mid (x_1, \ldots, a, \ldots, x_n) \in X \}$$

The slice is naturally embedded in $\mathbb{R}^n$ as it can be thought of as the graph of a function with $n-1$ inputs, (namely, $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$) and one output (i.e. $f(x_1, \ldots, a, \ldots x_n)$)

E.g.



As for the dimension of the slice, let's first think about the dimension of the graph itself.

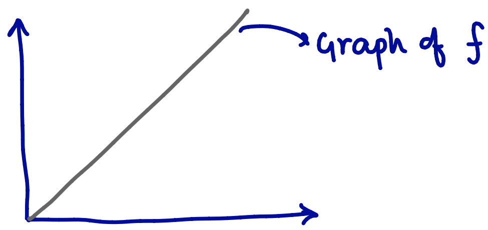
Assume also that f is a continuous function.

I claim that while the graph resides in a space of higher dimension, its dimension itself is the same as that of the domain. One way to think about this is that in the graph

$$\Gamma = \{ (x_1, \dots, x_n, f(\vec{x})) \in \mathbb{R}^{n+1} \mid (x_1, \dots, x_n) \in X \},$$

the only variables to vary are the input variables. The output variable is then completely determined and therefore doesn't add a "degree of freedom" and therefore doesn't change the dimension.

For e.g. Consider the graph of the function $f: \mathbb{R} \to \mathbb{R}$ where $f(x) = x$. It is drawn as follows



If an ant were to live on the graph itself, the grey line, it can only move forward or backward, therefore it perceives only 1-dimension, same as the input space.

To prove more rigorously, consider the map:

φ_1: Domain of $f = X \longrightarrow \Gamma =$ Graph of f

$(x_1, \dots, x_n) \longmapsto (x_1, \dots, x_n, f(\vec{x}))$

This is a continuous map as each of the component maps is continuous, i.e.,

$(x_1, \dots, x_n) \mapsto x_1$

$(x_1, \dots, x_n) \mapsto x_2$

$\vdots$

$(x_1, \dots, x_n) \mapsto x_n$

$(x_1, \dots, x_n) \mapsto f(\vec{x})$ are all continuous. (δ-ε definition can also be applied here)

This map has an inverse, given as follows:

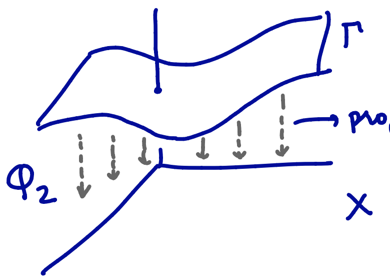
φ_2: $(x_1, \dots, x_n) \longleftarrow (x_1, \dots, x_n, f(\vec{x}))$

also continuous because continuous in each coordinate.

This map φ_2 is really a projection map.

Therefore X & Γ can be deformed onto each other and the process can be reversed, so intuitively, they have the same 'dimension'.

E.g.



project down from Γ to X & continuously deform the 'sheet' that corresponds to Γ into the 'sheet' that corresponds to X.

The process can be reversed via φ_1.

Now, when we take $(x_i = a)$ slice of the graph, as argued earlier, we can think of it as a graph on an input space where x_i is fixed. Therefore, the dimension of the slice of the graph is the dimension of the subset of X corresponding to $x_i = a$ (as that is our new input space).

If $X = \mathbb{R}^n$, setting $x_i = a$ reduces dimension of the input space by 1 & dimension of the slice of the graph is $n-1$.

If $X \neq \mathbb{R}^n$, and if there is no point in X that has the i^{th} coordinate a, then we lose no dimension!

Therefore, it would seem that one may lose upto 1 dimension upon restricting to the slice. Of course, we haven't yet defined dimension, so there is a certain amount of appeal to intuition going on & that's ok!

Finally, how is the slice situated with respect to the graph? Well, the slice is a subset of the graph. If f is continuous, it is in continuation with the rest of the graph.